LOCATIONS OF OUT-OF-PLANE EQUILIBRIUM POINTS IN THE ELLIPTIC RESTRICTED THREE-BODY PROBLEM UNDER RADIATION AND OBLATENESS EFFECTS

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ABSTRACT

This study deals with the generalization of the Elliptic Restricted Three-Body Problem (ER3BP) by considering the effects of radiation and oblate spheroid primaries. This may illustrate a gas giant exoplanet orbiting its host star with eccentric orbit. In the three dimensional case, this generalization may possess two additional equilibrium points (L₆,₇, out-of-plane). We determine the existence of L₆,₇ in ER3BP under the effects of radiation (bigger primary) and oblateness (small primary). We analytically derive the locations of L₆,₇ and assume initial approximations of (μ₁ ± √3A₂), where μ and A₂ are the mass parameter and oblateness factor, respectively. The fixed locations are then determined. Our results show that the locations of L₆,₇ are periodic and affected by A₂ and the radiation factor (q₁).

Key words: elliptic restricted three-body problem: out-of-plane: oblate spheroid: radiation

1. INTRODUCTION

The Classical Restricted Three-Body Problem (R3BP) has been generalized to include additional effects. The generalization of R3BP yields equilibrium points located out of the orbital plane (out-of-plane, L₆,₇) (Douskos & Markellos, 2006). L₆,₇ also have been found in the generalized Elliptic Restricted Three-Body Problem (ER3BP) (Singh & Umar, 2012). We study the locations of L₆,₇ in the generalized ER3BP when considering effects of radiation only in the bigger primary (radiation factor: q₁ ≠ 1, q₂ = 1) and an oblate spheroid only for the smaller primary (oblateness factor: A₁ = 0, A₂ ≠ 0). This study is applicable for the determination of L₆,₇ locations in exoplanetary systems which have a gas giant planet.

2. EQUATIONS OF MOTION

Let m₁ and m₂ be the masses of the primaries (m₁ > m₂). In the rotational frame, the coordinates of the third infinitesimal object are (ξ, η, ζ), and of the primaries are (ξ₁, η₁, ζ₁), and (ξ₂, η₂, ζ₂). The unit of time is chosen to make the gravitational constant G = 1. We use qᵢ and Aᵢ to represent the radiation and oblateness coefficients of the bigger (i = 1) and smaller (i = 2) primaries, respectively, such that

\[ A_i = \frac{AE_i^2 - AP_i^2}{5R_i^2}, \quad q_i = 1 - \frac{F_{p_i}}{F_{g_i}}, \]

with 0 < A₁ < 1 and 0 < 1 - q₁ < 1. AEᵢ and APᵢ are the dimensional equatorial and polar radii, Rᵢ is the effective radius, and Fᵢ and F₡ are the radiation pressure and gravitational attraction forces. In this work we consider the case where only the bigger primary has a source of radiation (q₁ ≠ 1, q₂ = 1) and the smaller primary is an oblate spheroid body (A₁ = 0, A₂ ≠ 0). The equations of motion of the third object in a barycentric and dimensionless coordinate system (\( \xi = \frac{\xi}{r}, \eta = \frac{\eta}{r}, \zeta = \frac{\zeta}{r} \)) due to the combined effect of oblateness and radiation are

\[ \ddot{\xi} - 2\dot{\eta} = V_\xi, \quad \ddot{\eta} + 2\ddot{\xi} = V_\eta, \quad \ddot{\zeta} = V_\zeta, \quad (1) \]

where

\[ V = (1 + \cos f)^{-1} \left[ \frac{1}{2} \left( \ddot{\xi}^2 + \ddot{\eta}^2 - 2f \ddot{\xi} \ddot{\eta} \cos f \right) + \frac{1}{r_1^2} \left( q_1(1 - \mu) + \frac{\mu}{r_2^2} \left( 1 + \frac{A_2}{2r_2^2} \left( \frac{3\zeta^2}{r_2^2} \right) \right) \right) \right], \]

and

\[ \mu = m_2/(m_1 + m_2) \leq \frac{1}{2}. \]
By substituting Equation 2 into Equation 1 and considering period and affected by $A_2$ and $q_1$. Our study suggests that the increasing value of $A_2$ produces the location of $L_{6,7}$ that are farther from orbital plane, while reducing $q_1$ produces locations closer to the orbital plane.

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**REFERENCES**


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**Figure 1.** Relation between $f$ and the position of $L_6$ ($\xi_6$ (left), $\zeta_6$ (right)). $\mu = 0.3$, $A_2 = 0.02$, $q_1 = 0.99$, and $e = 0.1$

$\begin{align*}
\eta &= \sqrt{1 + \frac{3A_2}{2}}, \\
n_1 &= \sqrt{(\xi - \mu)^2 + \eta^2 + \zeta^2}, \\
n_2 &= \sqrt{(\xi - \mu + 1)^2 + \eta^2 + \zeta^2},
\end{align*}$

where $f$ is true anomaly, $e$ is the eccentricity, and $r$ is the distance between primaries. $\xi$, $\eta$, $\zeta$, $\xi_0$, $\eta_0$, and $\zeta_0$ denote the partial derivatives with respect to $f$.

**3. LOCATION OF OUT-OF-PLANE POINTS**

The locations of $L_{6,7}$ can be obtained by imposing conditions as follow:

$$\begin{align*}
\dot{\xi} &= \dot{\eta} = \dot{\zeta} = 0, \\
\dot{\xi}_0 &= \dot{\eta}_0 = 0, \\
\dot{\zeta}_0 &= 0.
\end{align*}$$

By substituting Equation 2 into Equation 1 and considering $n_1^2 = 1 + 3A_2$ and $n_2^2 = 3A_2$ we obtain

$$\begin{align*}
\xi_0 &= \frac{(\mu - 1)}{n^2} \left[ \frac{\mu}{3\sqrt{3}A_2^2} + \frac{q_1}{3\sqrt{3}A_2^2} \right], \\
\eta_0 &= \pm \sqrt{3A_2} \left[ \frac{\mu}{3\sqrt{3}A_2^2} + \frac{q_1}{3\sqrt{3}A_2^2} \right], \\
\zeta_0 &= -\frac{6\sqrt{3}A_2^{3/2}(\mu - 1)q_1}{5\mu(3A_2^2 + 1)^{3/2}} + 1 \right]^{1/2},
\end{align*}$$

with $(\xi_0, +\eta_0)$ the position of $L_6$ and $(\xi_0, -\eta_0)$ the position of $L_7$. We determine that the existence of $f$ in Equation 3 makes the locations of $L_{6,7}$ in ER3BP periodic. Equation 3 also shows that the locations of $L_{6,7}$ are affected by $q_1$ and $A_2$. An increasing value of $A_2$ produces locations of $L_{6,7}$ that are farther from the orbital plane, while reducing $q_1$ produces locations closer to the orbital plane.

**Figure 2.** Position of $L_6$ ($\xi_6$, $\zeta_6$) as a function of $f$, with $\mu = 0.3$, $q_1 = 0.99$, $e = 0.1$, and $A_2$: 0.0198 (•), 0.0199 (●), and 0.0200 (●).

**Figure 3.** The position of $L_6$ ($\xi_0$, $\zeta_0$) as a function of $f$, with $\mu = 0.3$, $q_1 = 0.99$, $e = 0.1$, and $A_2$: 0.0198 (•), 0.0199 (●), and 0.0200 (●).